In terms of the coordinates ρ and θ, this relation is

$$\frac{\partial\psi}{\partial\lambda} = \frac{\partial\psi}{\partial\rho}\frac{\partial\rho}{\partial\lambda} + \frac{\partial\psi}{\partial(\rho\theta)}\frac{\partial(\rho\theta)}{\partial\lambda} \tag{26}$$

The gradients of ρ and θ with respect to λ are known and given by the expressions $\partial\rho/\partial\lambda = \cos\alpha$, and $\partial(\rho\theta)/\partial\lambda = \sin\alpha$, where α is the angle between λ and the ρ coordinate line. When these relations are substituted into Eq. (26) and the resulting expression maximized with respect to α, there results the equation

$$\left.\frac{\partial\psi}{\partial\lambda}\right|_{\max} = \left[\left(\frac{\partial\psi}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2\right]^{1/2} \tag{27}$$

which can be used with Eqs. (24, 25, and 27) to yield the maximum shear stress:

$$\tau_{\max} = \gamma A(t)\left[\frac{(\partial\psi/\partial\rho)^2 + (1/\rho^2)(\partial\psi/\partial\theta)^2}{|\omega'(\zeta)|^2}\right]^{1/2}$$

The displacement function ψ has been evaluated for the six-point, star-perforated shape shown in Fig. 1, which has been mapped with a 50-term mapping function. The original ballistic configuration is indicated by the dashed lines. The curvilinear coordinates are shown over a part of the cross section, demonstrating the slight mismatch between the true circular outer boundary and the curvilinear coordinate. A contour map of the displacement is shown in Fig. 2 for a step-function acceleration load, i.e., $A(t) = gH(t)$.

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Projectile Ranges in Excess of 180°

Richard V. Elms Jr.*
Lockheed Missiles and Space Company, Sunnyvale, Calif.

Expressions are derived for the maximum range of a projectile where the range exceeds 180° in vacuum on a nonrotating spherical earth. The expressions are in terms of the burnout altitude and of the ratio of kinetic-to-potential energy at burnout. The corresponding angle between the burnout velocity vector and the local vertical is also determined. The results are presented graphically, and the limiting conditions are discussed.

I. Introduction

Blitzer and Wheelon[1] derived expressions for the maximum range of a projectile in vacuum on a nonrotating spherical earth and for the burnout angle corresponding to this range. The solution offered was in terms of a dimensionless parameter, the ratio of the kinetic to potential energy of the projectile at launch. This paper extends the meaning of maximum range to investigate trajectories in vacuum which result in impacts on a nonrotating spherical earth greater than 180° from the burnout point. Obviously, minimum energy criteria would rule out ballistic trajectories for ranges in excess of 180°. Such ranges, however, are of interest in range safety considerations for satellite launches and may be of some significance in ballistic missile flights where payload and accuracy would be sacrificed to avoid countermeasures along the shorter path. The method presented by Blitzer and Wheelon is used in this investigation.

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* Senior Research Engineer.

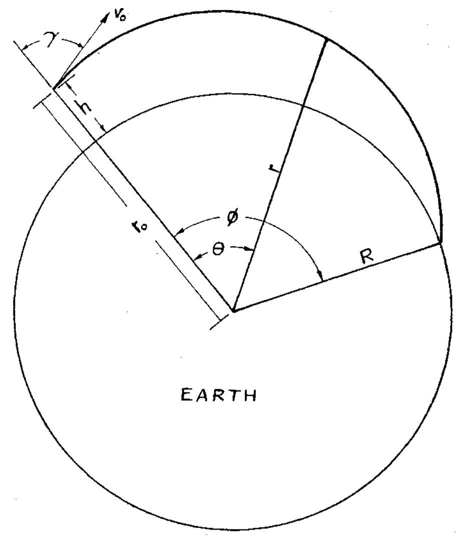

Fig. 1 Coordinate system.

The equations of motion for a particle moving in an inverse-square force field are[2]

$$\ddot{r} - r\dot{\theta}^2 + (GM/r^2) = 0 \tag{1}$$

$$(d/dt)(r^2\dot{\theta}) = 0 \tag{2}$$

where G is the constant of gravitation, and M is the mass of the earth. The initial conditions are (see Fig. 1):

$$\begin{aligned} r &= r_0 = R + h & \dot{r} &= \dot{r}_0 = v_0\cos\gamma \\ \theta &= \theta_0 = 0 & \dot{\theta} &= \dot{\theta}_0 = (v_0/r_0)\sin\gamma \end{aligned} \tag{3}$$

where v_0 is the burnout velocity, and γ is the burnout angle (the angle between the burnout velocity vector and the local vertical). A form of the solution, using Eqs. (1–3), is the familiar hit equation:

$$\frac{R+h}{R} = \frac{\sin(\gamma - \phi)}{\sin\gamma} + \frac{GM(1 - \cos\phi)}{(R+h)v_0^2\sin^2\gamma} \tag{4}$$

where h is the burnout altitude, R is the radius of the earth, and ϕ is the (angular) range from the burnout point to the first intersection of the earth's surface by the projectile's orbit. The dimensionless parameter $\alpha \equiv (R+h)v_0^2/GM$ is twice the ratio of kinetic-to-potential energy at burnout. Substituting α in Eq. (4), one has

$$\frac{R+h}{R} = \frac{\sin(\gamma - \phi)}{\sin\gamma} + \frac{(1 - \cos\phi)}{\alpha\sin^2\gamma} \tag{5}$$

II. Maximum Range Conditions

The minimum energy required to attain the range $\phi = 180°$ is expended for a given burnout altitude when $\gamma = 90°$ (orbit 1, Fig. 2). For the same burnout altitude h and holding $\gamma = 90°$, any increase in v_0 (increase in α) results in a new orbit with no intersection of the earth's surface (orbit 2, Fig. 2, e.g.). If the h and v_0 of orbit 2 are held constant while γ is varied, orbits that intersect the earth's surface can be produced. If γ is sufficiently *increased* from the initial 90° value, an orbit tangent to earth's surface will result. Further increases in γ produce orbits that intersect the earth's surface. The forementioned tangent and intersection points will be less than 180° from the burnout point ($\phi < 180°$). This may be seen from the fact that the perigee of an orbit is less than 180° about the earth's center in the direction of motion from any point on the orbit where the velocity vector is directed below the local horizontal. The forementioned tangent point is the perigee of an orbit.

If γ is sufficiently *decreased* from the value of 90°, an orbit that is tangent to the earth's surface at $\phi > 180°$ is produced (orbit 3, Fig. 2). This may be seen from the fact that the apogee of an orbit is less than 180° about the earth's center in the direction of motion from any point on the orbit where the velocity vector is directed above the local horizontal. The tangent point is the perigee of an orbit and is, of course, 180° from the apogee. Continuing with the h and α values of the tangent orbit 3 (same values as orbit 2), a further decrease in γ reduces the range (orbit 4, Fig. 2, e.g.), and when $\gamma = 0°$, the impact point will be under the burnout point ($\phi = 0°$). Reversing the process, one can increase γ from $\gamma = 0°$ until the tangent orbit is again obtained at $\phi > 180$. A further increase in γ will produce an orbit that does not intersect the earth's surface, and, at $\gamma = 90°$, orbit 2 (Fig. 2) will be obtained.

The following conclusions can be drawn from this discussion. 1) Projectile ranges in excess of 180° from the burnout point require more energy than the minimum energy required for a 180° impact range.[1] This also means that projectiles that can impact at $\phi > 180°$ have sufficient energy to be in orbits that do not intersect the earth's surface and would be so with the correct values of burnout angle γ. 2) Projectile ranges in excess of 180° can be obtained only when $\gamma \leq 90°$ ($\gamma = 90°$ are the special tangent orbits, where $h = 0$, and $\phi = 360°$). 3) The maximum range ($\phi > 180°$), for a fixed h and α, occurs when the tangent orbit is produced, and the condition on ϕ and γ is $d\phi/d\gamma = \infty$.

Looking again at orbit 3, Fig. 2, one may hold h and γ for this tangent orbit constant and vary v_0. Since $\alpha = (R + h)v_0^2/GM$, this is the same as varying α. Increasing α from the orbit 3 value will produce orbits that do not intersect the earth's surface; decreasing α will decrease the range until, at $\alpha = 0$, $\phi = 0$. Reversing the process and increasing α from zero, while holding the h and γ of orbit 3, Fig. 2 constant, will increase ϕ until the tangent case (orbit 3) is obtained. A further increase in α beyond the tangent orbit value will, as before, produce orbits that do not intersect the earth's surface. Thus the maximum range in excess of 180° for a fixed h and γ occurs with the tangent orbit, and, at this point, $d\phi/d\alpha = \infty$.

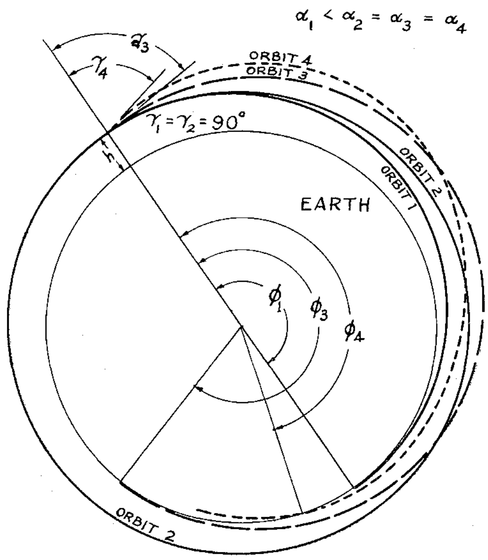

Fig. 2 Long range orbits.

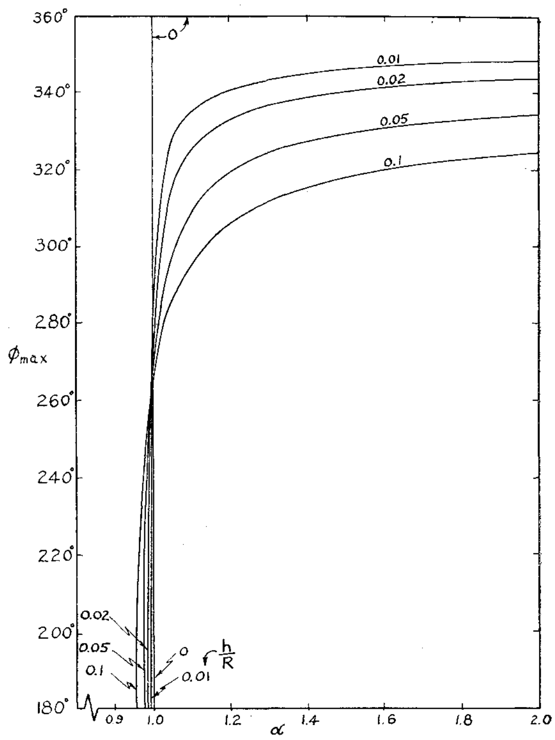

Fig. 3 Maximum range vs α.

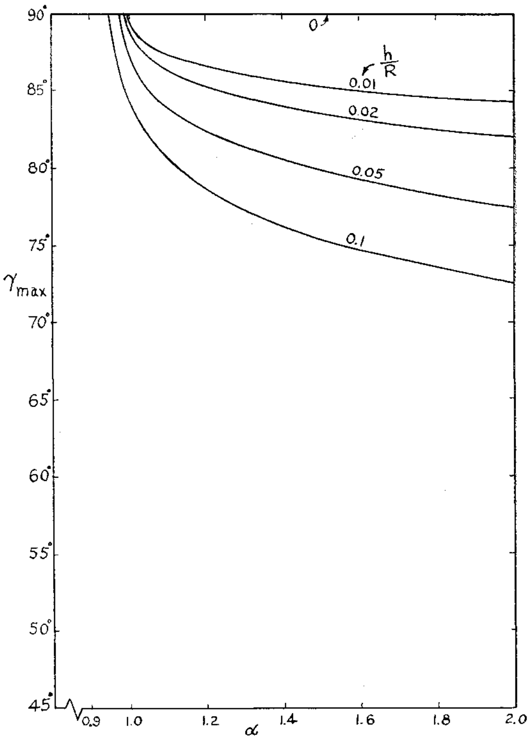

Fig. 4 Burnout angle for maximum range.

III. Maximum Range

Holding α and h constant, differentiate each term in Eq. (5), solve for $d\phi/d\gamma$, and, then,

$$\frac{d\phi}{d\gamma} = \frac{2(1 - \cos\phi)\cot\gamma - \alpha\sin\phi}{\sin\phi - \alpha\sin\gamma\cos(\gamma - \phi)}$$

Setting $d\phi/d\gamma = \infty$, a maximum range condition for $\phi > 180°$, gives

$$\sin\phi_{max} - \alpha \sin\gamma_{max} \cos(\gamma - \phi)_{max} = 0 \quad (6a)$$

or

$$\cot\phi_{max} = \frac{1 + \cot^2\gamma_{max} - \alpha}{\alpha \cot\gamma_{max}} \quad (6b)$$

where γ_{max} is the burnout velocity angle for maximum range. The explicit expression for ϕ_{max} is obtained by combining Eqs. (6) with the basic hit equation (5). Equation (5) is manipulated to form a term that is the right side of Eq. (6b), and $\cot\phi_{max}$ is substituted:

$$\cot^2\gamma_{max} = \frac{h^2 \sin^2\phi_{max}}{R^2(1 - \cos\phi_{max})^2} \quad (7)$$

Since $\gamma \leq 90°$ when $\pi \leq \phi \leqslant 2\pi$, $\cot\gamma$ is positive, and

$$\tan\gamma_{max} = -(R/h) \tan(\phi_{max}/2) \qquad (\text{for } \phi_{max} > 180°) \quad (8)$$

Equation (7) is now combined with the basic hit equation (5):

$$\phi_{max} = \cos^{-1}\left[\frac{\alpha - (h/R)^2 - 1}{\alpha(1 + h/R) + (h/R)^2 - 1}\right] \quad (9)$$

Differentiating the basic hit equation (5) to obtain $d\phi/d\alpha$, while holding h and γ constant, gives

$$\frac{d\phi}{d\alpha} = \frac{1 - \cos\phi}{\alpha \sin\phi - \alpha^2 \sin^2\gamma \sin\phi - \alpha^2 \sin^2\gamma \cot\gamma \cos\phi}$$

When one sets $d\phi/d\alpha = \infty$, a maximum range condition for $\phi > 180°$ is obtained:

$$\cot\phi_{max} = \frac{1 + \cot^2\gamma_{max} - \alpha}{\alpha\cot\gamma_{max}} \quad (10)$$

which is identical with Eqs. (6) and leads to Eq. (9).

IV. Discussion

The maximum range ϕ_{max} and the corresponding launch angle are plotted in Figs. 3 and 4 as functions of α for various values of h/R. It can be shown that for $\alpha < 2$ the orbit is elliptical and therefore closed, for $\alpha = 2$ the orbit, is parabolic, and for $\alpha > 2$ the orbit is hyperbolic. In the practical sense, $\phi_{max} > 180°$ requires a closed orbit and, since parabolic and hyperbolic orbits are not closed, $\alpha < 2$ is the maximum value of α considered. This, then, limits ϕ_{max} for a given burnout altitude other than $h = 0$.

It appears that the curves in Fig. 3 have a common point of intersection at $\alpha = 1.0$, but the use of Eq. (9) will show that this is not the case. As shown in Fig. 3, the larger ranges beyond $\phi_{max} = 270°$ for a given α require small burnout altitudes. For the burnout altitudes considered, ranges between 180 and 270° require relatively small increases in α over the α_{min} value for $\phi_{max} = 180°$. To provide an idea of the energy required for extreme impact ranges, which the ratio α does not give, the energy per pound of a satellite in a circular orbit at 100-naut-mile alt is assumed to be the energy per pound which a projectile has at a burnout altitude of $0.02R$. The burnout α is then 1.01. Similarly, the satellite energy per pound for circular orbits of 200, 500, and 1000 naut miles, when given to a projectile at a $0.02R$ burnout altitude, result in burnout α's of 1.04, 1.11, and 1.21, respectively.

A circular orbit results when $\alpha = 1$ and $\gamma = 90°$. If $h = 0$, the orbit just touches the round model earth everywhere on the orbit. With $h = 0$, $\gamma = 90°$, and $1 < \alpha < 2$, the orbits are tangent to the earth at the launch point, and $\phi_{max} = 360°$ as indicated in Fig. 3.

The conversion of ϕ_{max} and γ_{max}, which are in inertial coordinates, to ϕ_{max} and γ_{max} in earth coordinates will require a determination of the influence of the earth's rotation. This influence will depend on the launch point latitude and the burnout velocity azimuth.

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Observations on Minor Circle Turns

PAUL D. ARTHUR* AND BRUCE E. BAXTER†

Lockheed-California Company, Burbank, Calif.

The equatorial tangent minor circle turn is studied in detail as a generalization and correction of some earlier work by Jackson. The turn described herein provides convenient closed-form solutions for studying lateral maneuvering. It is demonstrated that the turn can be described as a simple geometric relationship in terms of orbital elements by a single parameter *Q*. Position in the turn is then shown to be a function of *L/D* and the available kinetic energy. The bank-angle schedule to fly the turn results in an easily programmed function of velocity.

Nomenclature

m = mass
g = constant of proportionality between weight and mass
V = velocity
r = radius from center of earth to vehicle
L = lift
D = drag
Q = equatorial minor circle parameter
N = defined by Eq. (25a)
i = orbit inclination
φ = latitude
λ = longitude
ψ = heading angle measured from a latitude line
γ = flight path angle
β = bank angle measured from local vertical
η = $(V/V_c)^2$
α = defined by Eq. (23)

Subscripts

i = initial
f = final
c = circular

A MINOR circle turn requires a vehicle to fly in a minor circle rather than in its natural great circle trajectory and necessitates continuous aerodynamic bank control to maintain the flight path. The minor circle may be defined in several ways: Loh[1] prefers to specify the turn by the colatitude, whereas Shaver[2] specifies the turn by the latitude. The polar circles thus defined describe trajectories along a line of constant latitude and give range in terms of the arc length along the turn. Jackson[3] has additionally studied closed-form solutions of lateral maneuvering which give range in terms of orbital elements. An integration error in

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* Consultant, Spacecraft Organization. Associate Fellow Member AIAA.

† Senior Thermodynamics Engineer, Spacecraft Organization. Member AIAA.